

# NONCOMMUTATIVE GEOMETRY AND A CLASS OF COMPLETELY INTEGRABLE MODELS

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## Abstract

We introduce a Hodge operator in a framework of noncommutative geometry. The complete integrability of 2-dimensional classical harmonic maps into groups ( $\sigma$ -models or principal chiral models) is then extended to a class of ‘noncommutative’ harmonic maps into matrix algebras.

## 1 Introduction

A generalization of classical (pseudo-) Riemannian geometries is obtained by generalizing the concept of differential forms, accompanied with a suitable generalization of the Hodge operator. The algebra of (ordinary) differential forms is replaced by a differential algebra on some, in general noncommutative, algebra. In this setting one can consider ‘noncommutative’ analogues of physical models and dynamical systems. After collecting some basic definitions in section 2, section 3 presents a class of completely integrable generalized harmonic maps into matrix algebras which are noncommutative analogues of harmonic maps into groups (also known as  $\sigma$ -models or principal chiral models). This extends our previous work [1-4] where, in particular, the nonlinear Toda lattice has been recovered as a generalized harmonic map with respect to a ‘noncommutative differential calculus’ on  $\mathbb{R} \times \mathbb{Z}$ . Section 4 contains some conclusions.

## 2 Basic definitions

### 2.1 Differential calculus over associative algebras

Let  $\mathcal{A}$  be an associative algebra over  $\mathbb{C}$  (or  $\mathbb{R}$ ) with unit element  $\mathbf{1}$ . A *differential algebra* is a  $\mathbb{Z}$ -graded associative algebra (over  $\mathbb{C}$ , respectively  $\mathbb{R}$ )  $\Omega(\mathcal{A}) = \bigoplus_{r \geq 0} \Omega^r(\mathcal{A})$  where the spaces  $\Omega^r(\mathcal{A})$  are  $\mathcal{A}$ -bimodules and  $\Omega^0(\mathcal{A}) = \mathcal{A}$ . A

differential calculus over  $\mathcal{A}$  consists of a differential algebra  $\Omega(\mathcal{A})$  and a linear<sup>1</sup> map  $d : \Omega^r(\mathcal{A}) \rightarrow \Omega^{r+1}(\mathcal{A})$  with the properties

$$d^2 = 0, \quad d(w w') = (dw) w' + (-1)^r w dw' \quad (1)$$

where  $w \in \Omega^r(\mathcal{A})$  and  $w' \in \Omega(\mathcal{A})$ . The last relation is known as the (generalized) *Leibniz rule*. We also require  $\mathbb{1}w = w\mathbb{1} = w$  for all elements  $w \in \Omega(\mathcal{A})$ . The identity  $\mathbb{1}\mathbb{1} = \mathbb{1}$  then implies  $d\mathbb{1} = 0$ . Furthermore, it is assumed that  $d$  generates the spaces  $\Omega^r(\mathcal{A})$  for  $r > 0$  in the sense that  $\Omega^r(\mathcal{A}) = \mathcal{A} d\Omega^{r-1}(\mathcal{A}) \mathcal{A}$ .

## 2.2 Hodge operators on noncommutative algebras

Let  $\mathcal{A}$  be an associative algebra with unit  $\mathbb{1}$  and an involution  $^\dagger$ . Let  $\Omega(\mathcal{A})$  be a differential calculus over  $\mathcal{A}$  such that there exists an invertible map  $\star : \Omega^r(\mathcal{A}) \rightarrow \Omega^{n-r}(\mathcal{A})$  for some  $n \in \mathbb{N}$ ,  $r = 0, \dots, n$ , with the property

$$\star(wf) = f^\dagger \star w \quad \forall w \in \Omega^r(\mathcal{A}), f \in \mathcal{A}. \quad (2)$$

Such a map  $\star$  is called a (generalized) *Hodge operator*.<sup>2</sup> We will furthermore assume that  $^\dagger$  extends to an involution of  $\Omega(\mathcal{A})$  so that

$$(w w')^\dagger = w'^\dagger w^\dagger. \quad (3)$$

Then the further condition

$$(\star w)^\dagger = \star^{-1}(w^\dagger) \quad (4)$$

can be consistently imposed on the calculus, since

$$(\star(wf))^\dagger = (f^\dagger \star w)^\dagger = (\star w)^\dagger f = [\star^{-1}(w^\dagger)] f = \star^{-1}(f^\dagger w^\dagger) = \star^{-1}[(wf)^\dagger]. \quad (5)$$

We still have to define how the exterior derivative  $d$  interacts with the involution. Here we adopt the rule<sup>3</sup>

$$(dw)^\dagger = (-1)^{r+1} d(w^\dagger) \quad w \in \Omega^r(\mathcal{A}). \quad (6)$$

## 2.3 Noncommutative harmonic maps into matrix algebras

Let  $\mathcal{A}$  be an associative algebra with unit  $\mathbb{1}$  and  $\mathcal{H}$  an algebra generated by the entries  $\mathbf{a}^i_j \in \mathcal{A}$ ,  $i, j = 1 \dots, N$ , of a matrix  $\mathbf{a}$  with generalized inverse<sup>4</sup>  $S$ , i.e.,

$$S(\mathbf{a}^i_k) \mathbf{a}^k_j = \delta_j^i \mathbb{1} = \mathbf{a}^i_k S(\mathbf{a}^k_j). \quad (7)$$

<sup>1</sup>Here *linear* means linear over  $\mathbb{C}$ , respectively  $\mathbb{R}$ .

<sup>2</sup>As a consequence, the inner product  $\Omega^1(\mathcal{A}) \times \Omega^1(\mathcal{A}) \rightarrow \mathbb{C}$  defined by  $(\alpha, \beta) = \star^{-1}(\alpha \star \beta)$  satisfies  $(\alpha, \beta f) = (\alpha f^\dagger, \beta)$  and  $(f \alpha, \beta) = (\alpha, \beta) f^\dagger$ .

<sup>3</sup>See also [5]. A different though equivalent extension of an involution on  $\mathcal{A}$  to  $\Omega(\mathcal{A})$  was chosen in [6]:  $(w w')^* = (-1)^{rs} w'^* w^*$  where  $w \in \Omega^r(\mathcal{A})$ ,  $w' \in \Omega^s(\mathcal{A})$ , and  $(dw)^* = d(w^*)$ . The two extensions are related by  $w^* = (-1)^{r(r+1)/2} w^\dagger$ .

<sup>4</sup>Examples are given by matrix Hopf algebras (cf [6]) in which case the antipode provides us with a generalized inverse.

Given a differential calculus  $(\Omega(\mathcal{A}), d)$ , the matrix of 1-forms

$$A := S(\mathbf{a}) d\mathbf{a} \quad (8)$$

satisfies the (zero curvature) identity

$$F := dA + AA = 0 . \quad (9)$$

Let us now assume that  $(\Omega(\mathcal{A}), d)$  admits a Hodge operator  $\star$ . The equation

$$d \star A = 0 \quad (10)$$

then defines a *generalized harmonic map* into a matrix algebra.<sup>5</sup>

A *conserved current* of a generalized harmonic map is a 1-form  $J$  which satisfies  $d \star J = 0$  as a consequence of (10). We call a generalized harmonic map *completely integrable* if there is an infinite set of independent<sup>6</sup> conserved currents.

### 3 Completely integrable 2-dimensional generalized harmonic maps

For 2-dimensional classical  $\sigma$ -models there is a construction of an infinite tower of conserved currents [7]. This has been generalized in [1-4] to harmonic maps on ordinary (topological) spaces, but with *noncommutative* differential calculi, and values in a matrix group. In the following, we present another generalization to harmonic maps on, in general *noncommutative* algebras (see also [4]).

Let  $(\Omega(\mathcal{A}), d)$  be a differential calculus over an associative algebra  $\mathcal{A}$  with unit  $\mathbb{1}$ , involution  $^\dagger$  and a Hodge operator  $\star$  satisfying the rules listed in section 2.2 with  $n = 2$ . Furthermore, let us consider a generalized harmonic map into a matrix algebra. If in addition the following conditions are satisfied, then the construction of an infinite tower of conservation laws (for classical  $\sigma$ -models) mentioned above also works in the generalized setting under consideration.<sup>7</sup>

1. For each  $r = 0, 1, 2$  there is a constant  $\epsilon_r \neq 0$  such that

$$\star \star w = \epsilon_r w \quad \forall w \in \Omega^r . \quad (11)$$

Using  $\star \star (\star w) = \star (\star \star w)$  we find

$$\epsilon_{2-r} = \epsilon_r^\dagger . \quad (12)$$

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<sup>5</sup>We may also call this a *generalized principal chiral model* or a *generalized  $\sigma$ -model*. Actually, we only need the restriction of the Hodge operator to 1-forms here, i.e.,  $\star : \Omega^1(\mathcal{A}) \rightarrow \Omega^{n-1}(\mathcal{A})$ .

<sup>6</sup>A convenient notion of independence in this context still has to be found.

<sup>7</sup>For *commutative* algebras, less restrictive conditions were given in [1,2].

(11) together with (4) in the form  $\star\star(\star\star w)^\dagger = w^\dagger$  leads to

$$\epsilon_r^\dagger = \epsilon_r^{-1} . \quad (13)$$

In particular, it follows that  $\epsilon_2 = \epsilon_0^\dagger = \epsilon_0^{-1}$  and  $\epsilon_1 = \pm 1$ .

2. We impose the modified symmetry condition<sup>8</sup>

$$(\alpha \star \beta)^\dagger = \epsilon_0 \beta \star \alpha \quad (14)$$

where  $\alpha, \beta \in \Omega^1(\mathcal{A})$ . This is consistent with (2) since

$$[\alpha \star (\beta f)]^\dagger = [\alpha f^\dagger \star \beta]^\dagger = \epsilon_0 \beta \star (\alpha f^\dagger) = \epsilon_0 (\beta f) \star \alpha . \quad (15)$$

3.  $\epsilon_0 = -\epsilon_1$ .

4. The first cohomology is trivial, i.e., for  $\alpha \in \Omega^1(\mathcal{A})$  we have

$$d\alpha = 0 \quad \Rightarrow \quad \exists \chi \in \mathcal{A} : \quad \alpha = d\chi . \quad (16)$$

As a consequence of (10),

$$J^{(1)} := D\chi^{(0)} = (d + A)\chi^{(0)} = A \quad \text{where} \quad \chi^{(0)} := \text{diag}(\mathbb{1}, \dots, \mathbb{1}) \quad (17)$$

is conserved. Let  $J^{(m)}$  be any conserved current. Using (11) and (16), this implies

$$J^{(m)} = \star d(\chi^{(m)\dagger}) \quad (18)$$

with an  $N \times N$  matrix  $\chi^{(m)}$ . Now

$$J^{(m+1)} := D\chi^{(m)} \quad (19)$$

is also conserved, since

$$\begin{aligned} d \star J^{(m+1)} &= d \star D\chi^{(m)} = -\epsilon_1 [D \star d(\chi^{(1)\dagger})]^\dagger = -\epsilon_1 [DJ^{(m)}]^\dagger \\ &= -\epsilon_1 [DD\chi^{(m-1)}]^\dagger = -\epsilon_1 [F\chi^{(m-1)}]^\dagger = 0 . \end{aligned} \quad (20)$$

The second equality in the last equation follows from the next result.

*Lemma.* For a matrix  $\chi$  with entries in  $\mathcal{A}$  we have

$$d \star D\chi = -\epsilon_1 (D \star d(\chi^\dagger))^\dagger . \quad (21)$$

*Proof:* First we note that

$$(d \star d\chi_j^i)^\dagger = d(\star d\chi_j^i)^\dagger = d \star^{-1} (d\chi_j^i)^\dagger = -d \star^{-1} d(\chi_j^i)^\dagger = -\epsilon_1 d \star d(\chi_j^i)^\dagger$$

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<sup>8</sup>As a consequence, the inner product defined in a previous footnote satisfies  $(\alpha, \beta)^\dagger = (\beta, \alpha)$ .

using (6), (4), again (6), then (11) and (13). Furthermore,

$$[d(\chi^k_j)^\dagger \star A^i_k]^\dagger = \epsilon_0 A^i_k \star d(\chi^k_j)^\dagger$$

using (14). Hence

$$\begin{aligned} d \star D\chi^i_j &= d \star (d\chi^i_j + A^i_k \chi^k_j) = d \star d\chi^i_j + d((\chi^k_j)^\dagger \star A^i_k) \\ &= d \star d\chi^i_j + d(\chi^k_j)^\dagger \star A^i_k + (\chi^k_j)^\dagger d \star A^i_k \\ &= [(d \star d\chi^i_j)^\dagger + (d(\chi^k_j)^\dagger \star A^i_k)^\dagger]^\dagger \\ &= [-\epsilon_1 d \star d(\chi^i_j)^\dagger + \epsilon_0 A^i_k \star d(\chi^k_j)^\dagger]^\dagger \end{aligned}$$

using  $d \star A = 0$ . Inserting  $\epsilon_0 = -\epsilon_1$  now completes the proof.  $\square$

In this way we obtain an infinite set of (matrices of) conserved currents. Introducing  $\chi := \sum_m \lambda^m \chi^{(m)}$  with a parameter  $\lambda \in \mathbb{C}$ , (18) together with (19) leads to the linear equation

$$\star d(\chi^\dagger) = \lambda D\chi \quad (22)$$

(see also [2]). As a consequence,  $D \star d\chi^\dagger = \lambda D^2 \chi = \lambda F \chi$  and  $0 = (d \star D\chi)^\dagger = -\epsilon_1 D \star d\chi^\dagger + (d \star A)^\dagger \chi$ , from which the following integrability condition is obtained,

$$[(d \star A)^\dagger - \epsilon_1 \lambda F] \chi = 0. \quad (23)$$

If  $F = 0$ , which is solved by (8), then the harmonic map equation  $d \star A = 0$  results. Alternatively,  $d \star A = 0$  is solved by  $A = \star d(\phi^\dagger)$  with a matrix  $\phi$  with entries in  $\mathcal{A}$ . Then the integrability condition becomes

$$0 = F = d \star d\phi^\dagger + (\star d\phi^\dagger)(\star d\phi^\dagger) = d \star d\phi^\dagger - d\phi d\phi \quad (24)$$

using (14), (11) and  $\epsilon_0 \epsilon_1 = -1$ .

### 3.1 Examples

(1) Let  $\mathcal{A}$  be the Heisenberg algebra with the two generators  $q$  and  $p$  satisfying  $[q, p] = i\hbar$ . In the simplest differential calculus over  $\mathcal{A}$  we have  $[dq, f] = 0$  and  $[dp, f] = 0$  for all  $f \in \mathcal{A}$ . It follows that  $df = (\hat{\partial}_q f) dq + (\hat{\partial}_p f) dp$  where the generalized partial derivatives are given by

$$\hat{\partial}_q f := -\frac{1}{i\hbar} [p, f], \quad \hat{\partial}_p f := \frac{1}{i\hbar} [q, f]. \quad (25)$$

Acting with  $d$  on the above commutation relations for ‘functions’ and differentials, one obtains  $dq dq = 0$ ,  $dq dp + dp dq = 0$  and  $dp dp = 0$ . As an involution we choose hermitean conjugation with  $q^\dagger = q$ ,  $p^\dagger = p$ . A Hodge operator satisfying the conditions (4), (11) and (14) is determined by

$$\star 1 = dq dp, \quad \star dq = dp, \quad \star dp = dq, \quad \star (dq dp) = -1, \quad (26)$$

so that  $\epsilon_0 = \epsilon_2 = -1$  and  $\epsilon_1 = 1$ . Now we consider a generalized harmonic map with values in the group of unitary elements  $U$  of  $\mathcal{A}$  which satisfy  $U^\dagger U = \mathbb{1} = UU^\dagger$ . With

$$A = U^\dagger dU = -\frac{1}{i\hbar} (U^\dagger p U - p) dq + \frac{1}{i\hbar} (U^\dagger q U - q) dp \quad (27)$$

we get  $\star A = (i\hbar)^{-1} (U^\dagger p U - p) dp - (i\hbar)^{-1} (U^\dagger q U - q) dq$  and the harmonic map equation  $d\star A = 0$  becomes  $[p, U^\dagger p U] - [q, U^\dagger q U] = 0$ . In terms of  $P := U^\dagger p U$  and  $Q := U^\dagger q U$  this takes the form

$$[p, P] - [q, Q] = -i\hbar (\hat{\partial}_q P + \hat{\partial}_p Q) = 0. \quad (28)$$

On the level of formal power series in  $q$  and  $p$ , every closed 1-form is exact so that (16) holds. All required conditions are fulfilled in this example. From (22) one derives

$$d\chi = \lambda(1 - \lambda^2)^{-1} (\lambda A - \star A) \chi \quad (29)$$

using  $A^\dagger = A$ . Reading off components with respect to the basis  $\{dq, dp\}$  of  $\Omega^1(\mathcal{A})$  leads to  $\chi q = L\chi$  and  $\chi p = M\chi$  where

$$L := \frac{\lambda}{1 - \lambda^2} (\lambda^{-1} q - p - \lambda Q + P), \quad M := \frac{\lambda}{1 - \lambda^2} (-q + \lambda^{-1} p + Q - \lambda P). \quad (30)$$

The integrability condition is then  $[L, M] = i\hbar$ .

(2) Let  $\mathcal{A} = C^\infty(\mathbb{R}^2)$  with the (noncommutative) Moyal product [8]

$$f \star h = m \circ e^{(i\hbar/2)P} (f \otimes h) \quad (31)$$

where  $P := \partial_q \otimes \partial_p - \partial_p \otimes \partial_q$  in terms of real coordinates  $q$  and  $p$ , and  $m(f \otimes h) = fh$  for  $f, h \in \mathcal{A}$ . An involution is given by  $(f \star h)^\dagger = h^\dagger \star f^\dagger$  where  $^\dagger$  acts as complex conjugation on the functions. In terms of the real generators  $q$  and  $p$  of  $\mathcal{A}$ , the simplest differential calculus<sup>9</sup> is determined by  $dq \star f = f \star dq$  and  $dp \star f = f \star dp$  as in our first example. A Hodge operator is then given by

$$\star 1 = dq \star dp, \quad \star dq = dp, \quad \star dp = dq, \quad \star(dq \star dp) = -1. \quad (32)$$

The differential calculus has trivial cohomology and all required conditions are satisfied. (10) implies  $A = \star d(\phi^\dagger)$  with a matrix  $\phi$  with entries in  $\mathcal{A}$ . The harmonic map equation is then obtained by substituting this expression into the zero curvature condition (9). We obtain

$$(\partial_q^2 - \partial_p^2)\phi - m \circ e^{(i\hbar/2)P} P(\phi \otimes \phi) = 0 \quad (33)$$

which is

$$\square\phi - \partial_q\phi \star \partial_p\phi + \partial_p\phi \star \partial_q\phi = 0. \quad (34)$$

This is a deformation of a classical principal chiral model. For  $\phi \in \mathcal{A}$  it is a deformation of the wave equation.

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<sup>9</sup>Differential calculi on the Moyal algebra were also considered in [9].

## 4 Conclusions

We introduced Hodge operators and a class of harmonic maps on (suitable) differential calculi on associative algebras. Furthermore, we generalized a construction of an infinite tower of conserved currents from the classical framework of  $\sigma$ -models [7] to this framework of noncommutative geometry. It involves a drastic generalization of a notion of ‘complete integrability’. This is a very peculiar property of an equation and we have presented a constructive method to determine corresponding equations. Further elaboration of examples is necessary to clarify their significance, however.

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